

Does quantum chaos exist?

A quantum Lyapunov exponents approach.

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Abstract

We shortly review the progress in the domain of deterministic chaos for quantum dynamical systems. With the appropriately extended definition of quantum Lyapunov exponent we analyze various quantum dynamical maps. It is argued that, within Quantum Mechanics, irregular evolution for properly chosen observables can coexist with regular and predictable evolution of states.

I. INTRODUCTION.

The theory of deterministic chaos has a venerable history, dating back to Poincare celebrated work (see [50]). Here, deterministic chaos refers to deterministic development with chaotic outcome. Another way to say this is that from moment to moment the system is evolving in a deterministic way, i.e., the current state of a system depends on the previous state in a rigidly determined way. However, measurements made on the system do not allow the prediction of the state of the system even moderately far into the future. It was observed that whenever dynamical chaos was found, it was accompanied by nonlinearity (cf. [30]). Another observation made long time ago was that exponential sensibility in nonlinear *classical* systems can lead to exceedingly complicated dynamical behavior (cf. [37]). The characteristic features of such a behavior, e.g. deterministic unpredictability, positive Lyapunov exponents, are typical attributes of chaotic (deterministic) dynamical systems (cf. [30], [25]). We emphasize that the theory of such classical dynamical systems is a well developed subject and contains many (concrete) examples of chaotic systems. For instance, we can describe the dynamics of a classical system as the evolution in the phase space governed by the canonical equations of motion. Solutions of these equations define the family of trajectories on the phase space. The complex behavior of certain class of trajectories can be reflected by the positiveness of Lyapunov exponents and for some choices of the Hamiltonians one can find concrete examples of such chaotic systems (see [30]).

However, contemporary science has been founded on *quantum mechanics*. Therefore, it is quite natural to look for the quantum-mechanical description of (classically) chaotic systems (see e.g. [1], [11], [17], [28], [30], [38], [53], [55], [59], [57], [61]). To achieve this objective it is important to realize that the chaotic behavior arises from nonlinearities in the equations of motion. Let us remind that the Schrödinger equation of motion is a linear one. Furthermore,

the Koopman's construction, in the theory of abstract dynamical systems, leads to linear evolution operators (see e.g. [6]). On that basis, the long debate on the existence of quantum chaos leads to a conjecture that quantum mechanics suppresses chaotic behavior in such a way that there is only a room for "fingerprints" of chaos [12].

Our objective in this paper is to show that *quantum mechanics admits, in some special cases, certain features of chaotic behavior*. To understand the origin of this phenomenon let us remind that the dynamics of a classical system can be described either as the evolution in the phase-space by the conventional Hamilton's equations or by the Liouville equations. Now if we consider a system with some ergodic properties e.g. mixing, then we can find that the chaotic and unpredictable behavior of some observables coexists with a regular and predictable behavior of densities (so states). The time evolution of a quantum system is described either by Schrödinger equations (quantum counterpart of the Liouville picture) or by the Heisenberg equations (quantum counterpart of the Hamilton's picture). We shall indicate how these scheme may be used to explain the just mentioned coexistence now in the quantum mechanical description. In other words, we shall show that a kind of unpredictability can also exist for quantum maps. To study this phenomenon we shall use quantum Lyapunov exponents. To make this point comprehensible let us point out that we shall study the stability of dynamical maps without any analysis of the integrability of the corresponding equations of motion.

In Sections II and III we review the structure of classical mechanics and basic properties of (classical) chaotic systems respectively. This material is mostly well known; however, we emphasize the role of the picture in the description of the time development of systems. Section IV deals with the question of (non)linear lifting of dynamical maps. The objective here is to show the possibility of nonlinear lifting of such maps as well as its relation to certain algebraic structure. Section V provides a detailed exposition of quantum counterparts of the previously reviewed characteristic features of classical dynamical systems. Section VI contains the definition of quantum characteristic exponents and their basic properties while Section VII indicates how these exponents may be used to study stability of quantum dynamical systems. We close, Section VIII, with a brief discussion on the obtained new results. For the reader's convenience we include the Appendix with some basic " C^* -algebraic vocabulary".

II. THE STRUCTURE OF CLASSICAL MECHANICS.

We start with a review of the essential facts of classical mechanics. Let $\Gamma \equiv \{q_1, \dots, q_n, p_1, \dots, p_n\}$ be the phase space of some classical system. The Newtonian systems with (local) forces derivable from a potential form the important and large class of dynamical systems. The dynamics of this class is described by the Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i = 1, \dots, n \quad (1)$$

where q_i, p_i denote generalized coordinate and momentum respectively, and $H = H(q_1, \dots, q_n, p_1, \dots, p_n)$ is the Hamiltonian of the system. Let $\Omega \subset \Gamma$ be a subset. We denote by \mathcal{A} the set of all complex-valued smooth functions on Ω . \mathcal{A} is the set of classical observ-

ables of the considered system. The Hamilton's equations (1) lead to one parameter group of maps of the phase space into itself

$$G_t : \Gamma \rightarrow \Gamma \quad (2)$$

where $G_t\mu$, $\mu \in \Gamma$, is the solution of the Hamilton's equations with the initial condition $G_t\mu|_{t=0} = \mu_0$. The transformations G_t can be lifted on the level of observables in the following way

$$U_t : \mathcal{A} \rightarrow \mathcal{A} \quad (3)$$

where

$$U_t f(\mu) \equiv f_t(\mu) = f(G_t\mu) \quad (4)$$

for any $\mu \in \Gamma$. Obviously, U_t is the one parameter family of linear maps on \mathcal{A} . The prescription given by (4) is sometimes called the Koopman's construction. The next observation is

$$\frac{\partial f_t(\mu)}{\partial t} = \sum_i \frac{\partial f_t}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f_t}{\partial p_i} \frac{\partial H}{\partial q_i} \equiv \{H, f_t\} \quad (5)$$

Here $\{\cdot, \cdot\}$ denotes the Poisson bracket which can be also considered as a derivation on the abelian algebra \mathcal{A} of classical observables (cf. Appendix). In other words, if we equip \mathcal{A} with the structure of abelian C^* -algebra then the Poisson bracket determines one parameter group of automorphism of \mathcal{A} and this group of automorphism represents the Hamilton's evolution of observables. Another basic concept in the description of classical systems is that of a state. Namely, a state of a system is described by a (probability) distribution ϱ on Γ . In particular, a pure state is given by

$$\varrho(q_1, \dots, q_n, p_1, \dots, p_n) = \delta(q_1 - q_1^0) \cdot \dots \cdot \delta(q_n - q_n^0) \cdot \delta(p_1 - p_1^0) \cdot \dots \cdot \delta(p_n - p_n^0) \quad (6)$$

where $\delta(x)$ is the Dirac's δ -function, $(q_1, \dots, q_n, p_1, \dots, p_n)$ is an arbitrary point in Γ while the considered fixed point in Γ is denoted by $(q_1^0, \dots, q_n^0, p_1^0, \dots, p_n^0)$. We remind the reader that evolution in classical mechanics can be given in one of two pictures. The one just described is so called the Hamilton's picture, i.e. time evolution of an observable is given by (5) while a state is time independent: $\frac{d\varrho}{dt} = 0$. The second one, so called the Liouville picture determines the evolution of a state, i.e. the time development of a distribution ϱ . The relevant equations of motion are

$$\frac{df}{dt} = 0, \quad \frac{d\varrho_t}{dt} = -\{H, \varrho_t\}. \quad (7)$$

Obviously, on the set $\{\varrho - \text{an arbitrary distribution}\} \equiv \mathcal{S}$ of distributions (states) one can define maps V_t (in the same way as in (4))

$$V_t \varrho \equiv \varrho_t. \quad (8)$$

Clearly, $V_t(\alpha\varrho_1 + (1 - \alpha)\varrho_2) = \alpha V_t(\varrho_1) + (1 - \alpha)V_t(\varrho_2)$ for any $\alpha \in (0, 1)$. We remind that these pictures are equivalent in the sense

$$\langle f_t \rangle_{\varrho} \equiv \int_{\Gamma} f_t(\mu) \varrho(\mu) d\mu = \int_{\Gamma} f(\mu) \varrho_t(\mu) d\mu \equiv \langle f \rangle_{\varrho_t}. \quad (9)$$

Thus, the following picture is emerging. The time evolution in classical mechanics can be described by either linear (U_t) or convex (V_t) maps. Therefore, there does not exist a room for nonlinearity which is a necessary condition for a chaotic behavior. But we remind (cf. Introduction) that there are many examples of concrete models with chaotic behavior. To explain this apparent contradiction let us make an important observation. Although the Hamilton's and Liouville pictures are equivalent in the sense given by (9) the Hamilton's picture offers larger possibilities. Namely, one can pick up a coordinate (or momentum) as an observable. Then the equations of motion given by (5) are reducing to that of the form (1). It is important to realize that equations (1) are on the phase space Γ (or on its subset) and for some choice of the Hamiltonian $H(q, p)$ one can obtain a system of nonlinear differential equations. To understand this observation within the above presented structure of classical mechanics it is enough to note that to get the mentioned nonlinear equations *we did not use the linear lifting, i.e. the Koopman's construction. Therefore, for a particular choice of (classical) observables and the Hamiltonian, there exists a chance to get a nonlinearity, so also to get trajectories with unpredictable behavior.* There is another possibility to get a nonlinearity for classical systems. Namely, one can ask for a non-linear version of Koopman's construction. However, this is a more subtle point and we will discuss this question later on.

III. CHAOTIC (CLASSICAL) SYSTEMS.

Let us consider a system of differential equations given by (1) where for simplicity we put $n = 1$. Suppose $\mathbb{R} \ni t \mapsto x^i(t) \equiv (q^i(t), p^i(t)) \in \Gamma$, $i = 1, 2$ is a solution of (1) with initial conditions $x^i(0) = (q^i(0), p^i(0))$, $i = 1, 2$. Put $\Delta x(t) \equiv \text{dist}(x^1(t), x^2(t))$. Then one can ask the following question: what is behavior of the function $\mathbb{R} \ni t \mapsto \Delta x(t)$? We are interested in the class of systems such that the growth of $\Delta x(t)$ is of the exponential type. We remind that the exponential growth is measured by Lyapunov exponents which can be defined as the following limits:

i) for a discrete dynamical system:

$$\lambda^{\text{cl}}(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |D_x \tau^n(y)| \quad (10)$$

where $D_x \tau^n(y)$ denote the directional derivatives of τ composed with itself n times at a point x in a direction y (cf. [25]).

ii) for a continuous dynamical system:

$$\lambda^{\text{cl}}(x, y) = \lim_{t \rightarrow \infty} \frac{1}{t} \log |D_x \tau_t(y)| \quad (11)$$

where $D_x \tau_t(y)$ denotes, as above, the corresponding directional derivative.

Remark 1 *If we examine a time evolution τ_t of a point x depending on a parameter v , then*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \left| \frac{d}{dv} \tau_t(x(v)) \right| \quad (12)$$

also can be considered as a measure of sensitivity of evolution with respect to initial conditions (determined by v) provided that the limit in (12) exists. This limit can be also called Lyapunov exponent.

Both limits measure the average rate of exponential growth of separation of orbits which at time zero differ by a small vector. This property is used as a measure of the sensitivity of dynamical system with respect to initial conditions.

Definition 1 *A dynamical system described by differential equations of the type (1) is called regular (irregular) if $\lambda^{cl} \leq 0$ ($\lambda^{cl} > 0$ respectively).*

Lyapunov exponents are closely related with Kolmogorov-Sinai entropy \mathcal{E}_{K-S} which is another key quantity for analysis of classical dynamical systems. \mathcal{E}_{K-S} measures mixing properties of a system and its relation to Lyapunov exponents for smooth enough systems is given by (cf. [47], [48], [49])

$$\mathcal{E}_{K-S} = \sum_{k; \lambda_k \geq 0} \lambda_k \quad (13)$$

Thus, for a large class of dynamical systems, a positivity of Kolmogorov-Sinai entropy indicates the existence of positive Lyapunov exponents. We recall that for more general class of dynamical systems (that is satisfying weaker smooth conditions) one should replace equality in (13) by an inequality (see [36]).

Example 1 *Let us consider a dynamical system determined by the following differential equation*

$$\frac{d^2}{dt^2}x = \kappa^2 x, \quad \kappa \geq 0. \quad (14)$$

The solution of (14) can be written as $x(t) = x_1 \cosh(\kappa t) + x_2 \sinh(\kappa t)$ where x_1, x_2 are constants describing the initial conditions. Put, for simplicity, $x_2 = 0$ and assume that we have two initial conditions x_1^I and x_1^{II} such that $|x_1^I - x_1^{II}| = \epsilon$. The corresponding solutions will be denoted by $x^I(t)$ and $x^{II}(t)$ respectively. We note

$$\Delta x(0) \equiv |x^I(0) - x^{II}(0)| = |x_1^I - x_1^{II}| = \epsilon \quad (15)$$

while

$$\Delta x(t) = \epsilon \cosh(\kappa t) \approx_{t \rightarrow \infty} \frac{\epsilon}{2} e^{\kappa t} \quad (16)$$

Thus, the considered system exhibits the exponential growth of separation of trajectories, i.e. a sensitivity with respect to initial conditions. However, it is important to realize that it is difficult to consider this system as a chaotic one.

This example clearly indicates that a definition of chaotic system presents a delicate problem. In particular, in the above example, we have gotten a sensitivity with respect to initial conditions since we allow trajectories to spread to infinity. This motivates

Definition 2 *A (classical) dynamical system exhibits a chaotic behavior if*

- *it is defined on a compact subset of the phase space (or the algebra of observables is defined on a compact subset of the phase space),*
- *it has an irregular motion with some ergodic properties.*

Let us comment on this definition. Firstly, the assumption of compactness excludes a possibility of a system with trajectories going to infinity. In other words, an irregular evolution should have a kind of repeatability. Let us emphasize that all computer simulations of chaotic systems have this feature. Next, we wish to briefly comment on the assumption of ergodicity. In real dynamical systems there are coexisting regions of chaotic behavior with regions of regular motion. So we need a kind of ergodicity to get a system with overwhelming region of irregular motion.

We want to close this Section with the two remarks on the just given definition of chaotic system. Firstly, we want to point out that the discussed definition is not the only one possible. On the contrary, there are many different approaches to this topic with various definitions of chaos. Even for *semigroups of linear operators on a Banach space, there is an attempt to introduce a concept of chaos* (see the definition given by Deveney [22], see also [23]). However, we strongly believe that for real mechanical systems the presented definition is the proper choice as it reflects our intuition on sensitivity of dynamics. Secondly, we want to stress again that the Liouville picture seems to be not the best choice for doing a study of chaotic systems. Clearly, one can examine the basic features of the Peron-Frobenius operator. Nevertheless, we think that such the work can only give some insight for "shadows" of the real chaotic behavior.

IV. IS THE KOOPMAN'S CONSTRUCTION THE ONLY ONE POSSIBLE?

In the previous Section it was argued that in the study of chaotic systems we should restrict the class of dynamical systems to that defined on a compact subset of the phase space Γ . This implies that the algebra of observables associated with such the system is the set of all complex valued continuous functions defined on Ω where Ω is a compact subset of Γ . Furthermore, equations of motion for the system - the Hamilton's equations (1) should be restricted to Ω . But one can pose a question: what is the proper choice for Ω . If we consider a conservative system (such systems have the dynamics given by the Hamilton's equations) the most natural choice is the isoenergetic surface. So, in the remainder of this section we assume that Ω is a compact isoenergetic surface of some dynamical system. Consequently, Ω will be also considered as a compact topological space. We denote by $C(\Omega)$ the Banach space of all \mathbb{C} -valued continuous functions on Ω . $C(\Omega)$ with pointwise defined multiplication, complex conjugation and the supremum norm is an abelian unital C^* -algebra. By the way, we stress that *the compactness of Ω is "translated" in the algebraic language by the property of $C(\Omega)$ to be a unital algebra*. But now we have the following problem: $C(\Omega)$ has the very rich algebraic structure, in particular one can write a square. So the concept of nonlinear map on $C(\Omega)$ is perfectly well understood (cf. Example 2, subsection VC). On the contrary, the situation on Ω is much less clear. To see the problem let us consider, as an example, one dimensional harmonic oscillator. Then Ω in this case is just the ellipse. So, a natural question is emerging: *Is it possible, starting from Ω , to define such an algebraic structure \mathcal{W} which is compatible with that of $C(\Omega)$?* In the just mentioned example of a harmonic

oscillator, the ellipse can be naturally equipped only with topological and measure structures and these structures are quite enough for the standard Koopman's construction. However, the ellipse has not an algebraic structure and therefore it is impossible to study a nonlinear lifting in this setting. To clarify this question let us emphasize that, here, the nonlinear lifting is understood as the procedure leading from a nonlinear map defined on $\Omega \subset \Gamma$ to a nonlinear map defined on the set of all observables. To solve the problem we propose to use the construction which was called "the nonlinear version of the Koopman's construction" (see [43]). To describe the solution of our problem we need some preliminaries. Let us recall that a pure state φ on an abelian C^* -algebra \mathcal{A} (called also a character) is a linear, continuous, positive and normalized functional on \mathcal{A} such that $\varphi(ab) = \varphi(a)\varphi(b)$ for $a, b \in \mathcal{A}$ (cf. Appendix). Denote the set of all characters by \mathcal{P} . Let us add that by Gelfand-Naimark theorem we can identify an abelian C^* -algebra $\mathcal{A} \equiv C(\Omega)$ with the algebra of all continuous functions on \mathcal{P} where \mathcal{P} is equipped with the pointwise convergent topology. Moreover, one can establish one-to-one correspondence between $\Omega \ni x \mapsto \varphi \in \mathcal{P}$. So to keep notations of the previous sections we shall identify Ω with \mathcal{P} . Let \mathcal{W}_Ω denote the algebra of continuous functions on $C(\Omega)$ generated by the set $\mathcal{P}(=\Omega)$ of all characters of $C(\Omega)$. Elements of \mathcal{W}_Ω are of the form $\vartheta = W(\varphi_1, \dots, \varphi_n)$, where $n \in \mathbb{N}$, W is a polynomial of n commuting variables with complex coefficients, and $\varphi_1, \dots, \varphi_n \in \mathcal{P}$. Using the correspondence between the set Ω and the set of all characters (cf. [43], see also [31] for details of the theory of operator algebras) one has the following inclusion

$$\Omega \subset \mathcal{W}_\Omega. \quad (17)$$

Now, let us remind that the Gelfand transform is defined as a map

$$\hat{\cdot}: \mathcal{A} \rightarrow C(\mathcal{P}) \quad (18)$$

which to every $a \in \mathcal{A}$ assigns a function

$$\hat{a}(\varphi) \stackrel{\text{def}}{=} \varphi(a). \quad (19)$$

where φ is a character. Consequently, $C(\mathcal{P})$ can be considered as the image of \mathcal{A} under the Gelfand transform.

Let $a \in \mathcal{A}$. We define the following complex valued function \tilde{a} on \mathcal{W}_Ω :

$$\tilde{a}(W(\varphi_1, \dots, \varphi_n)) = W(\varphi_1(a), \dots, \varphi_n(a)). \quad (20)$$

It is easy to show that \tilde{a} is in fact an extension of \hat{a} over \mathcal{W}_Ω . Namely, in the case $n = 1$ in (20) one has

$$\tilde{a}(W(\varphi)) = W(\varphi(a)) = \varphi(W(a)) = \widehat{W(a)}(\varphi). \quad (21)$$

Here the second equality follows from multiplicativity of the character φ . Further, we shall consider dynamical maps on $\mathcal{W}_\mathcal{P}$ which are of the following form

$$\theta(W(\varphi_1, \dots, \varphi_n)) = (T(W))(\tau(\varphi_1), \dots, \tau(\varphi_n)), \quad (22)$$

where T is a polynomial of one variable, and $\tau: \mathcal{P} \rightarrow \mathcal{P}$ is a continuous map. Let us denote the set of all maps in this form by Θ . Now we are in position to give (cf. [43])

Theorem 1 *For an arbitrary map $\theta \in \Theta$ there exists, in general nonlinear, the well defined map*

$$C(\mathcal{P}) \ni \hat{a} \mapsto U_\theta \hat{a} \in C(\mathcal{P}) \quad (23)$$

and θ is uniquely determined by U_θ where

$$U_\theta = U_\tau \circ T. \quad (24)$$

U_τ is the operator defined by $(U_\tau f)(x) = f(\tau(x))$.

Thus, we got the affirmative answer to the question posed in the beginning of this Section. There is a possibility to introduce the algebraic structure \mathcal{W}_Ω in such a way that the multiplication in this structure is obtained from that of \mathcal{A} . Therefore, nonlinearity of a map on \mathcal{W}_Ω is compatible with the nonlinearity of the map on \mathcal{A} . This gives a possibility of nonlinear lifting of a map on \mathcal{W}_Ω to a map on \mathcal{A} .

To make the basic idea of the presented construction more clear let us again consider the example of one dimensional harmonic oscillator. Then, the set Ω is equal to the ellipse. \mathcal{W}_Ω can be considered as “an algebraic reconstruction” of the phase space Γ (from the singled out subset Ω) in such a way that its multiplication is compatible with the natural multiplication of the algebra of (classical) observables $C(\Omega)$. The main difficulty in carrying out this reconstruction is that there is still one question unanswered. *Namely, we do not know a direct relation between $\Gamma(\supseteq \Omega)$ and \mathcal{W}_Ω .* On the other hand, this nonlinear version of Koopman’s construction suggests rather strikingly the explanation why among the well known examples of chaotic systems the periodically perturbed ones play a prominent role. Such the systems, in general, are not conservative ones. Let us consider the one dimensional kicked harmonic oscillator; a nearly paradigm of chaotic system. Let Ω be again the isoenergetic surface of the harmonic oscillator. To build \mathcal{W}_Ω we should construct over each point of Ω a “fibre” which is a free ring of one variable. Then one can suppose that the free motion is determined by a map τ defined on the isoenergetic surface, i.e., $\tau : \Omega \rightarrow \Omega$. The Theorem 1 says that τ can be linearly lifted to a map on \mathcal{A} . Then a kick (perturbation) acts on the system. Within this picture, one may say that a kick corresponds to a motion along a fiber. Consequently, a kick is reflected on the set of observables as a nonlinear map (see Theorem 1). Thus, we have got the explanation saying that kicked systems can be related with the “algebraic” nonlinearity. We remind, a nonlinearity is the necessary condition for a chaotic behavior.

Having clarified the basic concepts for classical dynamical system we want to pass to the basic subject of the paper, namely to the problem of existence and description of quantum chaos.

V. QUANTUM MECHANICS

It was pointed out in Section II that the basic concepts for a description of a classical dynamical system are: algebra of observables, states, and prescription for the equation(s) of motion which can be formulated in one of two pictures (equivalent in the sense of equality (9)). To quantize the (classical) scheme it is enough to change the realization of the algebra of observables with the relevant modifications of description of state and evolution.

Therefore, we replace the algebra $C(\Omega)$ (where Ω is a compact subset of Γ) by a unital (non-abelian) C^* -algebra \mathcal{A} . We repeat: *the unitality of the algebra is the algebraic translation of the topological compactness property!* Then, the states are linear positive, normalized (so continuous) functionals on \mathcal{A} . The time evolution is described by quantum counterparts of the Liouville and Hamilton's pictures, i.e. by the Schrödinger and the Heisenberg picture respectively. Our objective in this Section is to convince the reader that the Heisenberg picture is the suitable framework for investigations of quantum chaotic problems.

A. The Schrödinger picture

1. Linear Schrödinger equation

This is the quantum counterpart of the Liouville picture for a pure state. The basic equation of motion is

$$i\frac{\hbar}{2\pi}\frac{d\Psi(t)}{dt} \equiv i\hbar\frac{d\Psi(t)}{dt} = H\Psi(t) \quad (25)$$

where $\Psi(t)$ denotes the wave function. Its solutions are of the form

$$\Psi(t) = e^{-\frac{i}{\hbar}Ht}\Psi(0) \quad (26)$$

Thus

$$\|\Psi(t) - \Psi'(t)\| = \|\Psi(0) - \Psi'(0)\| \quad (27)$$

We observe that the equality (27) implies that *there is not any growth of separation of orbits which at time zero differ by a small vector!* Thus there does not exist a room for a sensitivity on initial conditions. Consequently, there is no chance for positive Lyapunov exponents. *We want to emphasize that exactly the same situation we have for a classical dynamical system with dynamics given in the Liouville picture.* Not taking into account the last remark, the long debate on existence of quantum chaos has led to a conjecture that quantum mechanics suppresses chaotic behavior in such a way that there is only a room for some “shadows” of chaos. Adopting this point of view one could only study special features of a wavefunction which describes quantum counterpart of classically chaotic system and to investigate some properties of the correspondence between classically chaotic system and its quantum counterpart. In other words we have the quantum analogy of the search of chaotic signatures within the Liouville picture (cf. the last paragraph of Section III).

Let us briefly discuss a typical result in this direction. The most famous proposition concerning a characterization of special properties of quantized classically chaotic systems is given by the following conjecture: There is a correspondence between classical (regular or irregular) dynamical system and the energy level statistics of the Hamiltonian of the quantum counterpart of the considered system. Namely,

$$\text{regular (classical) system} \leftrightarrow \text{Poisson distribution} \quad (28)$$

$$\text{irregular (classical) system} \leftrightarrow \text{Wigner – Dyson distribution} \quad (29)$$

where the Poisson distribution is given by $P(\Delta E) = ae^{-b\Delta E}$, a, b are some positive constants while the Wigner-Dyson distribution is given by $P(\Delta E) = A|\Delta E|^\alpha e^{-\beta(\Delta E)^2}$, with A, α, β some positive constants. However, recently (see [18]) it was shown that there are infinitely many counterexamples to the above conjecture. Thus, although there are some computer experiments verifying this conjecture for some models it is difficult to say that there is a well defined correspondence between energy level statistic and the nature of evolution.

As a matter of fact the same picture is emerging from the analysis based on the other apparently basic features of the wave function of quantized (classically) chaotic system. To specify them let us list: scars, revivals, localization of wave function, level repulsion leading to regions of complicated avoiding crossing. To the best of our knowledge, there does not exist any rigorous proof of a correspondence between such features and quantum chaotic systems. Therefore, looking for an answer to the question whether there is “true” chaos in quantum mechanics we drop this way of investigation of the posed problem.

2. Non-linear Schrödinger equation

Let us consider the Schrödinger type equation (in general, nonlinear)

$$i\hbar \frac{d\psi}{dt} = H_0\psi + V(\psi) \quad (30)$$

where $\psi \in \mathcal{H}$, \mathcal{H} is a separable Hilbert space, H_0 is the free Hamiltonian (yielding a linear evolution), and $V(\psi)$ is a (nonlinear) potential which is defined below. In other words, we begin with a field-theoretic model where the states at a given time are represented by vectors in \mathcal{H} while the nonlinearity enters through their time-dependence, i.e. the nonlinearity is described by (30). To give the definition of $V(\cdot)$ let us denote by $\{e_k\}$ a basis (independent of t) in \mathcal{H} . Define

$$\mathcal{H} \ni \psi \mapsto U(\psi) = \{ \langle e_k, \psi \rangle \} \in \ell_2 \quad (31)$$

(ℓ_2 is the Hilbert space of sequences $\{\alpha_i\}$, α_i a complex number, such that $\sum_i |\alpha_i|^2 < \infty$.) Denote by U the unitary map from \mathcal{H} onto ℓ_2 . We assume that a function $v(\{\alpha_i\}) \stackrel{\text{def}}{=} \{v(\alpha_i)\}$ satisfies the following condition

$$\sum_i |v(\alpha_i)|^2 < \infty \quad \text{provided that} \quad \sum_i |\alpha_i|^2 < \infty \quad (32)$$

The nonlinear term $V(\psi)$ in (30) is defined by

$$V(\psi) = U^{-1}v(U(\psi)) \quad (33)$$

Let us remark that if one takes another basis, say $\{f_i\}$, in \mathcal{H} then (31), (32), and (33) give, in general, another nonlinear operator $V'(\psi)$ on \mathcal{H} . However, one has

$$W^{-1}V'(W\psi) = V(\psi) \quad (34)$$

where W is the unique unitary operator on \mathcal{H} such that $W(\{e_k\}) = \{f_k\}$. Thus the operator $V(\psi)$ is defined in a “covariant” way.

As examples of nonlinear potentials leading to nonlinear type Schrödinger evolution (30) we can mention $V_1(\psi) = \bar{\psi}\psi\psi$ (its Heisenberg counterpart has applications to molecular dynamics and nonlinear optics, [19] see also [45], [46]) or $V_2(\psi) = \log(\psi)$ for $\psi \in \mathcal{H}_0 \subset \mathcal{H}$ where the domain \mathcal{H}_0 of the (unbounded) operation $\log(\cdot)$ is determined by the condition (32), (such evolution describes frictional effects in dissipative systems, [35], [52]). Other examples of nonlinear Schrödinger equations can be found in ([24], [13], [34]).

However, *one may ask whether the equation (30) is of the fundamental nature or it is a kind of approximation.* Although there are some arguments that nonlinear Schrödinger type equations could be fundamental ones (see e.g. [24], see also the argument given in Subsection VB), this question is still unanswered. Namely, one can also consider this type of equation as a result of an assumption that a part of a larger quantum system was regarded as “heat bath”. The lack of convincing arguments can be also related to the question of the nature of frictional effects. If one admits the conjecture that the Newton’s dynamics is more general one than that of the Hamilton’s the answer to the just raised question could be affirmative, i.e. one can expect to find “genuine” nonlinear quantum maps. Let us add that, in such the case, even on the level of classical mechanics the Liouville picture based on the Hamilton’s dynamics would be not the most general prescription for the evolution in the sense that nonlinear corrections would be necessary ingredient of the equations of motion.

Nevertheless, having a nonlinear Schrödinger equation one can ask for a possibility of chaotic behavior. To answer this question one can proceed as follows. We begin with rewriting (30) in terms of Fourier coefficients, i.e. using the basis $\{e_k \equiv |k\rangle\}$ in \mathcal{H} and the formula (30) we arrive to

$$i\hbar \frac{d}{dt} \sum_k \langle k|\psi\rangle |k\rangle = \sum_k \langle k|H_0\psi\rangle |k\rangle + \sum_k \langle k|V(\psi)\rangle |k\rangle. \quad (35)$$

Now, just for simplicity, let us put $H_0 = 0$. Moreover, to give as simple as possible arguments we take a concrete form of the nonlinear potential. Thus, in the definition of $V(\cdot)$ let us put $v(\alpha) = \kappa\bar{\alpha}\sum_n \delta(t - nt_0) + \frac{\chi}{2}\bar{\alpha}\alpha\alpha$ where κ and χ are positive constants, t_0 is a fixed period of time. Such potential was designed to describe the nonlinear evolution in the Kerr medium with kicks where the kicks are defined by the first term in the formula for v , i.e. $\kappa\bar{\alpha}\sum_n \delta(t - nt_0)$ (for details see [45], [46]). Hence

$$i\hbar \frac{d}{dt} z_k = \kappa \bar{z}_k \sum_n \delta(t - nt_0) + \frac{\chi}{2} \bar{z}_k z_k z_k \quad k = 1, 2, \dots \quad (36)$$

where $z_k = \langle k|\psi\rangle$. Let us note that owing to the properties of Dirac’s δ , (36) can be considered as a system of two equations; the first one determines the dynamics in the period between kicks while the second one gives the parametric process describing kicks in the time $t_n = nt_0$, $n = 1, 2, \dots$ (cf. [46]). Clearly, (36) is the (standard) nonlinear differential equation and an application of classical methods leads to solutions of (36) which admit a positive Lyapunov exponent (cf. [45], [46]). In other words (30) leads to an example of nonlinear Schrödinger type equations with sensitive solutions with respect to initial conditions where the Lyapunov exponent, λ_s^q , is defined as $\lambda_s^q = \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|\delta\psi(t)\|}{\|\delta\psi(0)\|}$ and $\|\psi\|$ is expressed in the Fourier coefficient terms (cf. [19]). So we conclude that within the nonlinear type of Schrödinger evolution there is a room for a chaotic behavior.

3. Density matrix formalism

The genuine quantum counterpart of the Liouville picture is that given in terms of density matrices. So, let us consider the quantum Liouville equation, called the von Neumann equation, which is of the form

$$\dot{\varrho} = -\frac{i}{\hbar}[H, \varrho] \quad (37)$$

where H stands for the Hamiltonian of the considered system, ϱ is a density matrix while $[\cdot, \cdot]$ denotes the commutator. Clearly (37) is a linear differential equation and one can ask for nonlinear version of this equation. We recall that nonlinear equations for time evolution of density matrix are known and used in physics (cf. [3], [10], [54]). Furthermore, similarly as for the Schrödinger equation, the question concerning the nature of the equation (37) as well as its nonlinear corrections can be posed. Basically one has the same answers. Therefore we will not discuss this point here; let us pass to an illustrative example instead. Let us consider a dynamical system with dynamics given by Hartree - type evolution equation (cf. [2]):

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H(\rho_t), \rho_t] \quad (38)$$

$$\rho_t|_{t=0} = \rho_o, \quad (39)$$

where $H(\rho) = [Tr(\rho Q)]Q$ with $Q = Q^*$, and ϱ is a density matrix on a finite dimensional Hilbert space \mathcal{H} . We recall that this type of equation can be obtained in the mean field limit for interacting quantum systems. The explicit solution of (38) is given by

$$\Phi_t(\rho_o) = \exp\left[-\frac{i}{\hbar}Tr(Q\rho_o)Qt\right]\rho_o \exp\left[\frac{i}{\hbar}Tr(Q\rho_o)Qt\right]. \quad (40)$$

Clearly, the solution $\Phi(\cdot)$ of (38) is a nonlinear quantum map.

To summarize, within the quantum Liouville picture there are examples of nonlinear evolution equations. Consequently, this can be taken as a motivations for our investigations of stability of dynamics determined by such the equations.

B. The Heisenberg picture

We recall that this is the quantum counterpart of the Hamilton's picture (the picture from the classical mechanics, cf. Section II). The fundamental equation of motion is of the form

$$\frac{d}{dt}A = \frac{i}{\hbar}[H, A] \quad (41)$$

where A is an operator in the algebra of observables, while H stands for the Hamiltonian of a system. The solution of (41) can be written in the form

$$A \mapsto e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht} \quad (42)$$

Our first remark is that the Heisenberg formula for equations of motion is in harmonious relation with the very rich algebraic C^* structure of the matrix formulation of quantum mechanics. On the contrary, the Schrödinger picture does not have this property: a Hilbert space has only a linear space structure while the set of density matrices for a real physical system is a Banach $*$ -algebra. There is only one exception. Namely, for finite dimensional models of quantum mechanics, the set of trace class operators is equal to the set of all bounded operators. Therefore, for such particular models one can also use the C^* algebraic technique for an analysis of evolution of density matrices. The next remark is related to a possibility of getting nonlinear operator equations. To be more specific, let H be a Hamiltonian of a physical system such that $H = H(A, B)$ is a function of several noncommuting dynamical variables. Obviously, in general, the definition of a function of two (or more) noncommuting variables is not a clear stuff. However, in Quantum Mechanics there are many examples of Hamiltonians of this type which have arisen as a result of the quantization procedure. So, here, we can put for the concrete $H(A, B)$ the relevant Ansatz. Then, the solution of the Heisenberg equation of motion for A (the equation needs not be linear in A)

$$A \mapsto e^{\frac{i}{\hbar}H(A,B)t} A e^{-\frac{i}{\hbar}H(A,B)t} \quad (43)$$

does not have to be the linear function (in A). Clearly, we have not such a possibility for the Schrödinger evolution of pure states.

Let us consider the question of nonlinear dynamical maps in the Heisenberg picture in detail. As it was mentioned at the beginning of this Section the Heisenberg picture is quantum counterpart of the Hamilton's picture. Let us remind (cf. Section II) that the Hamilton's equations also lead to a one parameter group of linear maps on the algebra \mathcal{A} of observables where $\mathcal{A} \equiv C(\Omega)$. In particular, the Poisson bracket is nothing else than the derivation on \mathcal{A} (cf. Appendix). However, as *it was pointed out in remarks following (9) there is a possibility, for the very special choice of observables, to study of time evolution without the linear lifting*. Namely, the choice of $\{q\}$ and $\{p\}$ can lead to nonlinear differential equations with solutions given by nonlinear maps (even to the solutions which are sensitive with respect to initial conditions). Now let us turn to quantum mechanics. We know, e.g. from quantum optics, that the creation a^\dagger and the annihilation a operators can be treated as a substitution for q and p . Let us take this point of view and let us put in (42, 43) $A = a^\dagger$, $B = a$. Then, we can expect to obtain nonlinear dynamical maps for some choice of Hamiltonians. Consequently, we can expect (similarly as in classical mechanics) to find in this way nonlinear equations and maps with interesting (non)stability properties. We want to add that another way of introducing nonlinear maps in quantum physics will be studied in the next subsection.

We close this subsection with a brief discussion on the equivalence of Schrödinger and Heisenberg pictures. Similarly, as for the classical mechanics (see (9)) we have

$$\begin{aligned} \langle A \rangle_{av}(t) &\equiv \langle A \rangle_{\Psi(t)} = (\Psi(t), A\Psi(t)) = (e^{-\frac{i}{\hbar}H(A,B)t}\Psi(0), Ae^{-\frac{i}{\hbar}H(A,B)t}\Psi(0)) \\ &= (\Psi(0), e^{\frac{i}{\hbar}H(A,B)t}Ae^{-\frac{i}{\hbar}H(A,B)t}\Psi(0)) \equiv \langle A(t) \rangle_{\Psi(0)} \equiv \langle A(t) \rangle_{av} \end{aligned} \quad (44)$$

To get a better understanding of the nature of nonlinearities described by (43) let us replace A in

$$A \mapsto e^{\frac{i}{\hbar}H(A,B)t} A e^{-\frac{i}{\hbar}H(A,B)t} \quad (45)$$

by $\langle A \rangle_{av}$. This can be done, e.g. by performing some approximations in averaging procedures (cf. [27], [45]). In this way we can arrive to

$$\langle A \rangle_{av} \mapsto f(\langle A \rangle_{av}, t) \quad (46)$$

where $f(\cdot)$ is, in general, a nonlinear function. Thus, we can obtain the “classical” nonlinear maps and we can expect to find examples having sensitiveness with respect to initial conditions. In fact such examples were found, e.g. see ([27], [45]). Obviously, in our discussion of stability properties of quantum systems we want to avoid the obscure procedure leading to (46). *To this end we shall introduce quantum Lyapunov exponents* (cf. Section VI).

To finish our discussion on equivalence of both pictures we should say few words on the relation between the nonlinear Schrödinger equation (30) and the Heisenberg equation (41). Obviously, the prescription (44) based on bilinear form (\cdot, \cdot) (given by the scalar product) is not the relevant one. However, to pass from a differential equation for a state to an operator equation involving an observable we can use “*the second quantization rule*” (cf. [14], [19], see also [39]). To explain this idea let us restrict ourself to the model described by equations (35) and (36). Then, the equation of motion of the annihilation operator a_i (respectively the creation operator a_i^*) follows from the equation of motion (36) in which we replace the Fourier coefficients z_i and \bar{z}_i (so complex numbers) by the boson creation and annihilation operators a_i and a_i^* (so operators). Thus, we have

$$i\hbar \frac{d}{dt} a_j = \kappa a_j^* \sum_n \delta(t - nt_0) + \frac{\chi}{2} a_j^* a_j a_j \quad (47)$$

The remark following (36) is also applicable to this case. Let us observe that (47) follows from the Heisenberg equation of motion

$$i\hbar \frac{d}{dt} a_j = [H, a_j] \quad (48)$$

with $H = i\frac{\kappa}{2} \sum_j [(a_j^*)^2 - (a_j)^2] \sum_n \delta(t - nt_0) + \frac{\chi}{4} \sum_j a_j^* a_j^* a_j a_j$. Thus, the second quantization methods links nonlinear Schrödinger type equation with the undoubtedly fundamental equation of motion of the Heisenberg form. Moreover, the latter can lead to nonlinear operator-valued equations.

To summarize this subsection, we conclude that within the Heisenberg picture there is a room for quantum counterparts of “nonlinear” type evolution.

C. Are there many nonlinear quantum maps?

We have pointed in previous Sections that although nonlinear relations are not sufficient for chaos of classical systems, some form of nonlinearity is necessary for chaotic dynamics. We observed that the Heisenberg equations of motion as well as the (nonlinear) Schrödinger

equations allow some form of nonlinearities. Besides we would like to know whether a nonlinearity can arise from the “nonlinear” quantization of nonlinear systems. We remind that in the theory of classical chaos we frequently study the dynamical maps alone. This arises the question for a quantization such the dynamical maps without referring to Hamiltonian equations. In other words we are talking about a possibility of quantization of Newtonian systems. Working in this direction, we have shown (cf. Section IV) that there exists the nonlinear version of Koopman’s construction. Here, we indicate how this construction can be used to argue that the well known class of nonlinear completely positive maps defined on a C^* -algebra (see Section VI and the Appendix) may be of the “physical” nature; for another motivation to study nonlinear completely positive maps in quantum dynamical systems see [10].

Let again $\mathcal{A}_{abel} \equiv C(\Omega) \equiv C(\mathcal{P})$ be an abelian C^* -algebra with unit where Ω is a compact (Hausdorff) space (cf. Section IV). Within the algebraic structure \mathcal{W}_Ω we can consider the map $\varphi \mapsto \varphi^n$. We observe

$$\hat{a}(\varphi) = \tilde{a}(\varphi) \mapsto \tilde{a}(\varphi^n) = \widehat{a^n}(\varphi) \quad (49)$$

where $a \in \mathcal{A}_{abel}$. In other words, there exists the correspondence between the map $\varphi \mapsto \varphi^n$, $\varphi \in \Omega$, $\varphi^n \in \mathcal{W}_\Omega$ and the map $a \mapsto a^n$, $a, a^n \in \mathcal{A}_{abel}$. Consequently, if a subset Ω of the phase space is algebraized properly, i.e. it is endowed with the \mathcal{W}_Ω structure we are able to translate nonlinearities of maps defined on \mathcal{W}_Ω to that defined on \mathcal{A}_{abel} . Let us illustrate this point.

Example 2 *Let us define*

$$\mathcal{L}^d \hat{f} \equiv r \hat{f}(1 - \hat{f}) \quad (50)$$

where 1 stands for the identity function, $\hat{f}(\cdot) \in C(\mathcal{P})$ while r is a constant. We have

$$\mathcal{L}^d \hat{f}(\varphi) = (r \hat{f} - r \hat{f}^2)(\varphi) = r \varphi(f) - r \varphi(f^2) = r \varphi(f) - r \varphi^2(f) = \quad (51)$$

$$= (r(\varphi - \varphi^2))(f) \equiv (\mathcal{L}\varphi)(f) \quad (52)$$

where $\varphi \in \mathcal{P}$, $\hat{f} \in C(\mathcal{P})$ and finally f is the element of \mathcal{A} which defines the Gelfand transform \hat{f} . Consequently, our example demonstrates rather strikingly that the logistic map (one of the most famous “chaotic” maps) can be considered both in $\mathcal{W}_\mathcal{P}$ and $C(\mathcal{P})$ terms. Evidently, \mathcal{L}^d as well as \mathcal{L} are nonlinear maps.

Let us pass to non-abelian case, i.e. we replace \mathcal{A}_{abel} by \mathcal{A} , where \mathcal{A} is an arbitrary nonabelian C^* algebra. We remind that \mathcal{A} can be considered as a noncommutative analogue of the space of bounded continuous functions, i.e. bounded continuous functions over a “quantum plane”. Here, we are using the phrase “quantum plane” in the sense of Manin (see [44]). Let us repeat the argument given at the beginning of this subsection but now applied for the quantum plane. We can expect that the maps of the form

$$a \mapsto \sum_n c_n a^n \quad (53)$$

where $a \in \mathcal{A}$, c_n is a number, have arisen as a result of lifting nonlinear maps defined over the quantum plane (cf. Theorem 1, Section IV). In other words, *it is our hypothesis that the maps of the form (53), or more generally completely positive nonlinear maps on a C^* -algebra can be considered as a result of (nonlinear) lifting nonlinear maps defined on the quantum plane.* This would mean that the quantization of a Newtonian system could lead to genuine nonlinear quantum map.

VI. QUANTUM LYAPUNOV EXPONENTS.

In this Section we develop the theory of quantum Lyapunov exponents (cf. [40], [41]). Let \mathcal{A} be a C^* -algebra with the unit $\mathbf{1}$, $\tau : \mathcal{A} \rightarrow \mathcal{A}$ a quantum map, i.e.

1. τ is positive: $\tau(A^*A) \geq 0$ for all $A \in \mathcal{A}$,
2. $\tau(\mathbf{1}) = \mathbf{1}$.

The pair (\mathcal{A}, τ) consisting of a C^* -algebra \mathcal{A} and a quantum map τ will be called the quantum system. We propose the following generalization of the classical Lyapunov exponent:

Definition 3 *Let (\mathcal{A}, τ) be a quantum system where τ is differentiable (in the Fréchet sense) map. Let $D_x \tau^n(y)$ denote the directional derivatives of τ composed with itself n times at a point x in a direction y . Then, the limit*

$$\lambda^q(\tau; x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|(D_x \tau^n)(y)\| \quad (\equiv \lambda^q), \quad (54)$$

whenever exists, will be called the quantum exponent.

Let us comment on this definition. Firstly, it is clear that (54) can be generalized for continuous dynamical systems in an obvious way (see Section III). Secondly, one can study the following weaker version of quantum characteristic exponent:

$$\lambda_\mu^q(\tau; x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mu((D_x \tau^n)(y))| \quad (\equiv \lambda_\mu^q), \quad (55)$$

where μ is a state on \mathcal{A} . However, as we are basically interested in quantum chaos it is enough for us to look for the largest Lyapunov exponent. As λ^q plays such the role (see properties of quantum exponents) we shall restrict ourself to definition involving the norm. Thirdly, the noncommutative generalization (54) of characteristic exponent is not the only one possibility. Namely, within the Connes's noncommutative geometry, [21], the required differential structure can be introduced by derivations (see Appendix for definitions) $\{\delta\}$ of the algebra \mathcal{A} associated with a physical system. Then using basically the same idea (see [26]) one can define

$$\Lambda^q = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\delta \tau_t(x)\| \quad (56)$$

where $x \in \mathcal{D}(\delta)$ and τ_t stands for (continuous) dynamics of the system. Moreover, we should mention that the presented two examples of definitions of quantum Lyapunov exponents do

not exhausted all possibilities (cf. [53], [56], [57]). However, studying the properties of quantum exponents as well as looking for quantum counterparts of chaotic systems (see Section VII) we have found λ^q as the best suited. Finally, the slight modification of the definition of Lyapunov exponents described in Remark 1 (see Section III) can be also done for quantum exponents. In fact, we shall use such the modification in our examples (5) and (6) (see Section VII).

Let us turn to the question of existence of the limit in the definition of λ^q . In other words, to present a convincing argument that λ^q is well defined we should give an example of sufficient conditions implying the existence of the limit in (54). Let us assume:

1. τ is a completely, in general nonlinear, positive map (see Appendix for appropriate definitions). In fact, this is the most important assumption for the next theorem. Namely, this condition implies a smoothness of the quantum map τ ; even one can say that the complete positiveness implies a kind of Laurent expansion for τ , (see Appendix for details).
2. $\|\tau^l(0)\| \leq C_1$ for all $l \in N$ and some positive C_1 , and

$$\Theta_\tau = \{x \neq 0 : \|\tau^l(x)\| \leq C_2\|x\| + \|\tau^l(0)\|\} \neq \emptyset \quad (57)$$

for some positive C_2 and all $l \in N$.

3. Finally

$$\|D_x \tau^k(y)\| > C^k(x, y) \quad (58)$$

for some positive $C(x, y)$ and all large $k \in N$.

Under the above assumptions one can prove (cf. [41]):

Theorem 2 *Let $\tau : \mathcal{A} \rightarrow \mathcal{A}$ be a map such that the assumptions (1), (2), (3) are satisfied. The following limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|D_x \tau^n(y)\| \quad (59)$$

exists for $x \in \Theta_\tau$.

The just given theorem clearly shows that the notion of quantum characteristic exponent λ^q is well defined for a large class of dynamical systems. Therefore, having the well-established notion of quantum counterpart of Lyapunov exponents λ^q let us turn to the description its basic properties. We have proved (see [41], [42])

- $\lambda^q(x, y) = \lambda^q(x, ay)$ for $a \in \mathbf{R} \setminus \{0\}$.
- Note that since the map $y \rightarrow D_x \tau^n(y)$ is linear one, it is natural to set

$$\lambda^q(x, 0) = -\infty. \quad (60)$$

- Let $\lambda^q(x, y) > \lambda^q(x, z) > -\infty$ and, additionally, let τ satisfy assumption (iii) of the theorem in the direction $y + z$. Then:

$$\lambda^q(x, y + az) \leq \lambda^q(x, y), \quad (61)$$

where $a \in \mathbf{R}$.

- The function $y \rightarrow \lambda^q(x, y)$ as the limit of continuous functions (in y) is, in general, the Baire function of type I. In particular, the set $\{y | y \rightarrow \lambda^q(x, y) \text{ is a continuous function}\}$ is dense in \mathcal{A} .
- Let \mathcal{A} be the C^* -algebra generated by a fixed self-adjoint operator and identity on some Hilbert space, i.e. \mathcal{A} is an abelian C^* -algebra. Furthermore, let $\tau : \mathcal{A} \rightarrow \mathcal{A}$ be a smooth map while ϕ a state on \mathcal{A} . Note that such dynamical system $(\mathcal{A}, \tau, \phi)$ should be considered as the classical one. Then, if some mild technical conditions are met (cf. [42]) Definition 3 leads to the largest classical characteristic exponent.

It is important to observe that the above listed properties of λ^q are reminiscent to that of classical characteristic exponents (cf. [7], [49]). Therefore, *we conclude that the notion λ^q is the well defined quantum counterpart of (the largest) characteristic exponent.*

VII. EXAMPLES.

In this Section we want to study concrete models of dynamical systems. In particular, we are interested in examples of systems with irregular time evolution, i.e. with systems with positive quantum exponent. Moreover, we shall consider examples with analytically calculated exponents. Unless otherwise stated the existence of quantum Lyapunov exponent for each example was proved separately. Let us begin with the simplest model.

(1) A dynamical system composed of N -level quantum system with dynamics given by a linear dynamical semigroup S_t of contractions. The assumption of local nonexpansiveness is always satisfied but the condition of variability is satisfied for some directions. In general, $\lambda^q(x, y) \in (-\infty, 0)$. As an illustration let us consider $S_t x = e^{-\lambda t} x$, where $t \geq 0$ and $\lambda > 0$. Then $\lambda^q(x, \delta x) = -\lambda$. This means the stability of the considered semigroup dynamics.

(2) A dynamical system with dynamics given by Hartree - type evolution equation (cf. [2]):

$$\frac{d}{dt} \rho_t = -i[H(\rho_t), \rho_t] \quad (62)$$

$$\rho_t|_{t=0} = \rho_o, \quad (63)$$

where $H(\rho) = [Tr(\rho Q)]Q$ with $Q = Q^*$, and ρ is a density matrix on a finite dimensional Hilbert space \mathcal{H} . Let us recall that this type of equation can be obtained in the mean field limit for interacting quantum systems. The explicit solution of (62) is given by

$$\Phi_t(\rho_o) = \exp[-iTr(Q\rho_o)Qt]\rho_o \exp[iTr(Q\rho_o)Qt]. \quad (64)$$

Thus taking the discrete time in (64) and putting $\tau = \Phi_1$ in (54) we get:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \| -i[Tr(\delta\rho Q)Qn, \rho] + \delta\rho \| = 0. \quad (65)$$

In (65) $[\cdot, \cdot]$ denotes the commutator. Clearly, assumptions of our theorem are satisfied. As $\lambda^q = 0$, it follows that Hartree-type evolution (62) is an example of the regular motion.

(3) As the next example let us consider the dynamical system $(\mathcal{B}(\mathcal{H}), \Phi)$ where $\mathcal{B}(\mathcal{H})$ denotes all linear operators on a finite dimensional Hilbert space while the map Φ is given by

$$\Phi(\rho) = \rho^2 \quad (66)$$

for $\rho^* = \rho$ from unit ball in $B(\mathcal{H})$. We want to stress that this example is rather “pure” mathematical one in the sense that it is well defined mapping as well as the “second order” term in the expansion of completely positive nonlinear maps on C^* -algebra (cf. Appendix). But, it is difficult to provide a (physical) Hamiltonian leading to the considered map. On the other hand, let us recall that the “square” is the basic ingredient of the logistic map - the famous paradigm of (classical) chaotic systems. So, if one assumes that the nonlinear version of Koopman’s construction can be applied to nonlinear quantization (cf. Section IV and subsection Vc) one can expect to find the physical interpretation of this map. Finally we want to add that the transformation of this type was used by Gisin and his coworkers (cf. [9]) for the study of the theory of quantum computers. So one can conclude that such systems may be of great significance for a general theory.

Having the clear motivation let us turn to an examination of its properties. It is easy to observe (cf. Euler theorem) that

$$\|D_\rho \Phi^n(\rho)\| = 2^n \|\rho\|^{2^n-1} \quad (67)$$

for $\rho \neq 1$ and $\rho \notin \text{Projectors}(\mathcal{H})$. To prove (67), it is enough to note that $\|\rho\| = \sup_{\lambda \in \text{spectrum}(\rho)} |\lambda|$. Clearly, the assumptions of the theorem are satisfied. Consequently, (67) leads to:

$$\lambda^q(\rho, \rho) = \lim_{n \rightarrow \infty} \frac{1}{n} ((2^n - 1) \log \|\rho\| + n \log 2). \quad (68)$$

As a conclusion, this mathematical example shows that the quantum exponent λ^q can also be positive (put $\rho : \|\rho\| = 1$, then $\lambda^q(\rho, \rho) = \log 2 > 0$). Thus the quadratic map, in some regions, can exhibits the irregular form of the “motion”.

In the next three examples we reexamine dynamical systems which were semiclassically treated recently. Namely, using a kind of approximation (cf. our discussion following (45)), it was shown that these models of quantum origin display signatures of chaotic motion. Now we want to treat these systems in pure quantum way and compare results. We want to add that in these examples the Hilbert space \mathcal{H} of a system *is not assumed to be of finite dimension*.

(4) Let us consider a quantum two-level system interacting with a single mode of the electromagnetic field (cf. [27], [32]). The Hamiltonian for a such system can be taken as

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega(a^\dagger a + \frac{1}{2}) + \hbar\lambda_0\sigma_x(a + a^\dagger) \quad (69)$$

where σ_x, σ_z are Pauli matrices and a^\dagger and a stand for bose creation and annihilation operators. The energy separation of the two-levels is equal to $\hbar\omega_0$ while the frequency of the radiation mode is ω . The time evolution of the system is given by the following operator equation

$$\dot{A} = \frac{i}{\hbar}[H, A] \quad (70)$$

for an observable A , where the dot denotes a time derivative. Let us remark that the Heisenberg equation (70) gives for this model first order coupled nonlinear operator equations (cf. [27]):

$$\begin{aligned} \dot{\sigma}_x &= -\omega\sigma_y \\ \dot{\sigma}_y &= \omega_0\sigma_x - 2\lambda\sigma_z(a + a^\dagger) \\ \dot{\sigma}_z &= 2\lambda\sigma_y(a + a^\dagger) \\ (a + a^\dagger)^\cdot &= -i\omega(a - a^\dagger) \\ (a - a^\dagger)^\cdot &= -i\omega(a + a^\dagger) - 2i\lambda\sigma_x \end{aligned} \quad (71)$$

Although the Hamiltonian H depends, in general, on the time t , one can solve (70) and subsequently compute the quantum exponents λ^q (see [32]). In particular one has:

$$\lambda^q(\sigma_i, \sigma_j) = 0 \quad (72)$$

for $i, j = 1, 2$, where $\sigma_1 = \sigma_x$ and $\sigma_2 = \sigma_z$. Consequently, this quantum optic model exhibits quantum regular evolution. Let us add that using some approximations a signature of chaos was shown (cf. [27]).

(5) This example is the quantum counterpart of a parametrically kicked nonlinear oscillator (cf. [33]). Its hamiltonian H is of the form

$$H = \frac{\chi}{2}(a^\dagger)^2 a^2 + i\hbar\frac{\kappa}{2}[(a^\dagger)^2 - a^2] \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nt_0) \quad (73)$$

where χ and κ are constants characterizing the system, a^\dagger and a stand for bose creation and annihilation operators, and t_0 is the period of free evolution (i.e. the evolution described by $H_{NL} = \frac{\chi}{2}(a^\dagger)^2 a^2$). Let us remark that H_{NL} gives the nonlinear part of the evolution. Obviously, $\delta(\cdot)$ stands now for the Dirac distribution. Then, the Heisenberg equation of motion,

$$\dot{A} = \frac{i}{\hbar}[H, A] \quad (74)$$

can be solved in terms of self-adjoint operators Φ and Π , where Φ and Π are defined by: $a = \Phi + i\Pi$. The solution of the Heisenberg equation for the considered model can be written as:

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{\frac{-i\mu}{2}} \begin{pmatrix} e^r \cos \mu B_0 & e^r \sin \mu B_0 \\ -e^{-r} \sin \mu B_0 & e^{-r} \cos \mu B_0 \end{pmatrix} \begin{pmatrix} \Phi(t_{n-1}^+) \\ \Pi(t_{n-1}^+) \end{pmatrix} \quad (75)$$

where $\mu = \chi t_0$, r the effective constant for kicks, and $B_0 = a^\dagger a - \frac{1}{2} = (\Phi^2 + \Pi^2 - 1)$.

Let us remark that (75) gives a nonlinear evolution since the matrix on the right hand side of (75) is the nonlinear function of operators Φ and Π . We want to study the stability properties of the evolution (75). We restrict ourself to an examination of the Lyapunov instabilities of the quadrature components of the electric field during the time evolution given by the formula (75). To do this let us define the quadrature operators:

$$\Phi^\epsilon = \frac{1}{2}[e^{i\epsilon}a + e^{-i\epsilon}a^\dagger] \quad (76)$$

and

$$\Pi^\epsilon = \frac{1}{2i}[e^{i\epsilon}a - e^{-i\epsilon}a^\dagger] \quad (77)$$

where $\epsilon \in [0, 2\pi]$. Let us note that the operators Φ^ϵ and Π^ϵ are related to the amplitude components of electric field. At this point in order to avoid any confusion some explanation is necessary. Namely, the Hamiltonian (73) leads to *the nonlinear evolution map* (75). However, from physical point of view, *an examination of the stability properties of dynamical map does not allow any change of annihilation (so also creation) operators nor any modification of the fixed Hamiltonian*. This makes the big difference between the treatment of classical and quantum models. Namely, in classical models to study stability properties of dynamical maps on the phase space (for the special choice of observables, cf. Sections II and III) we can take a slightly different coordinate and momentum. Here, creation and annihilation operators play the role of coordinate and momentum. Again, we made a very particular choice of observables. The result was nonlinear operator-valued equations. Therefore, to find *physical examples with positive quantum exponents* λ^q and to keep the uniqueness of creation and annihilation operators we are examining the quadrature operators. In other words, the quadrature operators are examples of dynamical variables in our model which can be varied without any change of a (a^\dagger) and the Hamiltonian. Thus we shall study quantum exponents in the sense of Remark 1 (see Section III).

To compute quantum exponents for quadrature operators in the considered model we should find an explicit formula for the norm of $\|D_\epsilon \tau^n(y)\|$, where in the example considered, by τ we denote the time evolution of the system between two successive kicks as well as the effect of the first kick. The variable y stands for an element of the set $\{\Phi, \Pi\}$. Clearly, τ^n denotes τ composed with itself n times. D_ϵ is the derivative with respect to the phase angle ϵ which describes a “rotation” in (Φ, Π) variables and which is used in the definition of quadrature operators. But, since a^\dagger and a are unbounded operators, in order to compute $\|D_\epsilon \tau^n(\cdot)\|$ we should introduce $(\Phi_\delta, \Pi_\delta)$, the cut-off in the (Φ, Π) variables (see [33] for details). Then, replacing the original dynamics by its well defined approximation and

changing the variables (Φ, Π) to $(\tilde{\Phi}, \tilde{\Pi})$ one can find such set of parameters (χ, κ, t_0) that $\lambda^q(\tilde{\Pi}) > 0$. It is remarkable that this result does not depend on the performed cut-off. In other words, for large enough r (r is the effective constant for kicks) and some values of $\mu = t_0\kappa$, the quantum characteristic exponent λ^q for quantum variable $\tilde{\Pi}_\delta$ is strictly positive. To comment on this result let us note that in polyparametric cases physics as well as mathematics allow numerous combinations of stability in certain directions and irregularity in others. Consequently, in such the cases one can expect chaotic features of the evolution of the model for some values of χ, τ, r, κ and regularity for others. Here, we got a confirmation of such the behavior ($\lambda^q(\tilde{\Phi}_\delta)$ can be negative for some values of parameters κ, χ , and r and positive for other values of parameters). Moreover, let us add that the approach based on some approximations (cf. remark preceding example (4)) to the considered problem gives the similar result (see [45], [46]). We conclude that this model exhibits the hyperbolic type of instabilities.

(6) In the last example we consider the squeezed light in a nonlinear medium having the second-order susceptibility $(\chi^{(2)})$. An analysis of such system leads to the following equations (see [60], section 11):

$$\frac{d}{dz}a(z) = ka^\dagger(z), \quad (78)$$

$$\frac{d}{dz}a^\dagger(z) = \bar{k}a(z), \quad (79)$$

where $a(z)$ is the annihilation operator, k is the coupling constant and \bar{k} stands for the complex conjugation of k . The dependence of a on z is a result of the one dimensional propagation of the electric field along the z -axis. Obviously, the equations (78), (79) lead to

$$\frac{d^2}{dz^2}a(z) = |k|^2a(z). \quad (80)$$

It is easy to observe that the same equation can be derived from the Hamiltonian

$$H = \hbar\omega a^\dagger a + i\hbar\frac{\kappa}{2}(a^{\dagger 2} - a^2) \quad (81)$$

where $\omega = 0$. Such a Hamiltonian would appear if we take into account an oscillatory time dependence of a , and then eliminate the free evolution by the interaction picture. The solution to equations (78), (79) is

$$a(z) = \cosh(|k|z)a(0) + \frac{k}{|k|}\sinh(|k|z)a^\dagger(0). \quad (82)$$

It is clear that we can apply our definition of Lyapunov exponent λ^q to analysis of stability of the propagating squeezed light. The quadrature operators are taken in the form

$$P_\alpha(z) = \frac{1}{2}(e^{i\alpha}a(z) + e^{-i\alpha}a^\dagger(z)) \quad (83)$$

$$Q_\alpha(z) = \frac{1}{2i}(e^{i\alpha}a(z) - e^{-i\alpha}a^\dagger(z)) \quad (84)$$

where $\alpha \in [0, 2\pi]$. In particular, we have

$$P_\alpha(z) = \frac{1}{2} \left([e^{|k|z} \sin \alpha + e^{-|k|z} \cos \alpha] a(0) \right. \quad (85)$$

$$\left. + [-e^{|k|z} \sin \alpha + e^{-|k|z} \cos \alpha] a^\dagger(t) \right), \quad (86)$$

An easy calculation leads to

$$\lambda^q = \lim_{n \rightarrow \infty} \lim_{z \rightarrow \infty} \frac{1}{z} \ln \left\| \frac{dQ_\alpha^n(z)}{d\alpha} \right\| = |k| \geq 0 \quad (87)$$

where we put for simplicity $\frac{k}{|k|} = -1$, Q_α^n indicates the cut off in Q_α (since a and a^\dagger are unbounded operators we again have to apply cut-off).

It is remarkable, see (87), that this result does not depend on the cut-off. Hence, this is another quantum optic model with very unstable dynamics. It is important property of this model that the classical Lyapunov exponent calculated for the trajectory

$$t \mapsto \langle w | Q_\alpha(t) | w \rangle \quad (88)$$

has the same value as its quantum counterpart; in (88) w denotes a standard coherent state $a(0)|w\rangle = w|w\rangle$.

VIII. CONCLUSIONS.

We have shown that Quantum Mechanics allows some form of nonlinear equations for dynamical maps. Consequently, we have found a kind of coexistence of regular and irregular quantum motion; the coexistence resembling very much that of classical mechanics. Both forms of nonlinearity, in Classical and Quantum Mechanics, arose as a result of the special choice of observables ($\{p, q\}$ for classical systems, $\{a, a^\dagger\}$ for quantum systems). However we want to repeat that there is very important difference in the study of stability properties of the corresponding dynamical systems. Namely, studying a stability of classical maps for such the choice of dynamical variables one can consider an arbitrary (small) variation of p and q . Basically we can repeat this argument for quantum maps defined on the set of states (cf. examples 2 and 3, Section VII). *However, we can not do this for a and a^\dagger !* The proper way out of this situation is to study another class of observables, e.g. the quadrature operators (cf. examples 5 and 6, Section VII). Namely, although the quadrature operators are built directly from a and a^\dagger , additionally, they depend on a parameter (phase). Consequently, their evolution is described by nonlinear maps while their variation can be implemented by that of the parameter (phase). Clearly, this procedure does not effect the uniqueness of a , a^\dagger . To summarize we have shown that the Quantum Mechanics admits the irregularity of the evolution of some variables. We consider the positivity of λ^q - so irregular motion - as the basic signature of unstable dynamics. However, we do not identify this signature of chaos with the very chaos of dynamical system since to get a complete description of chaotic behavior of dynamical maps some additional geometrical assumptions on the space of dynamical variables as well as ergodic assumptions are necessary (cf. Section III).

The paper also deals with the related question of derivation of nonlinear quantum dynamical maps without resorting to quantization of equations of motion. We have shown the possibility for nonlinear lifting of dynamical maps. These investigations were intended as an attempt to motivate our interest in completely positive nonlinear maps on C^* -algebras. Our result suggests that there is a chance to obtain such the maps via “physical considerations”. However this question is at present far from been solved as we do not know the relation between the algebraic structure \mathcal{W}_Ω and the phase space Γ .

We want to end this paper with the remark that the presented results are steps toward a quantum ergodic theory, where a rigorous definition of quantum chaos may be established.

IX. APPENDIX.

The theory of C^* -algebras can be find in the books of [16], [31], [51] or [58]. Here we recall only few basic results which have been used in the paper.

- A Banach $*$ -algebra \mathcal{A} is called a C^* -algebra if it satisfies $\|a^*a\| = \|a\|^2$ for $a \in \mathcal{A}$.
- Let \mathcal{A} be an abelian C^* -algebra, i.e. $ab = ba$ for any $a, b \in \mathcal{A}$. A character φ , of \mathcal{A} , is a nonzero linear map, $\mathcal{A} \ni a \mapsto \varphi(a) \in \mathbb{C}$ such that

$$\varphi(ab) = \varphi(a)\varphi(b) \tag{89}$$

foe all $a, b \in \mathcal{A}$. The spectrum $\mathcal{P}(\mathcal{A}) \equiv \mathcal{P}$, of \mathcal{A} is to be the set of all characters on \mathcal{A} .

- If φ is a character of an abelian C^* -algebra \mathcal{A} then $\varphi(a) \in \sigma(a)$, the spectrum of (the operator) a for all $a \in \mathcal{A}$. Moreover, $|\varphi(a)| \leq \|a\|$ and $\varphi(a^*a) \geq 0$.
- (*Gelfand-Naimark theorem*) Let \mathcal{A} be an abelian C^* -algebra and \mathcal{P} the set of characters of \mathcal{A} equipped with the weak* topology inherited from the dual \mathcal{A}^* , of \mathcal{A} . It follows that \mathcal{P} is a locally compact Hausdorff space which is compact if and only if \mathcal{A} contains the identity. Moreover, \mathcal{A} is isomorphic to the algebra $C_0(\mathcal{P})$ of continuous functions over \mathcal{P} which vanish at infinity.
- A (symmetric) derivation δ is a linear map from a $*$ -subalgebra $\mathcal{D}(\delta)$, the domain of δ , into \mathcal{A} with the properties that
 1. $\delta(A)^* = \delta(A^*)$, $A \in \mathcal{D}(\delta)$
 2. $\delta(AB) = \delta(A)B + A\delta(B)$, $A, B \in \mathcal{D}(\delta)$
- *Theorem.* ([16]) Let \mathcal{A} be a C^* -algebra with identity $\mathbf{1}$, and let δ be a norm-densely defined norm-closed operator on \mathcal{A} with domain $\mathcal{D}(\delta)$. It follows that δ is the generator of strongly continuous one-parameter group of $*$ -automorphism of \mathcal{A} if and only if $\mathcal{D}(\delta)$ is a $*$ -algebra and δ is a symmetric derivation.

The listed properties of δ imply $\mathbf{1} \in \mathcal{D}(\delta)$ and $\delta(\mathbf{1}) = 0$.

- ([15]) Let again $\mathcal{A} = C(\Omega)$ be an abelian C^* -algebra. By the dimension of Ω we shall mean the Hausdorff dimension of Ω . The classification results for derivations of \mathcal{A} can be given in the following way:

1. In the case that $\dim \Omega = 0$ it turns out that all closed derivations are trivial (i.e. equal to 0) if Ω is total disconnected.
2. The case $\dim \Omega = 1$, $\mathcal{A} = C([0, 1])$ has the complete classification.
3. For dimension more than 2 only sporadic results are known.

These results clearly show that in our discussion of the Hamilton picture (cf. Section II) implicitly we have used some additional topological properties of the phase space Γ .

- Let \mathcal{A}_i , $i = 1, \dots, k$ and \mathcal{B} be C^* -algebras and $\mathcal{A}_1 \times \dots \times \mathcal{A}_k$ the Cartesian product of $\mathcal{A}_1, \dots, \mathcal{A}_k$. A map $\Phi : \mathcal{A}_1 \times \dots \times \mathcal{A}_k \rightarrow \mathcal{B}$ is said to be completely positive if, for every $n > 0$, the n -square \mathcal{B} -valued matrix $[\Phi(a_{1;p,r}, \dots, a_{k;p,r})]_{p,r=1}^n$ is positive whenever the n -square \mathcal{A}_i -valued matrices $[a_{i;p,r}]_{p,r}$ are positive for $i = 1, \dots, k$.
- *Theorem ([4])* Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a completely positive (in general, nonlinear) map between two C^* -algebras \mathcal{A} and \mathcal{B} . Then, there exist (uniquely) completely positive maps $\Phi_{m,n} : \mathcal{A} \rightarrow \mathcal{B}$ ($m, n = 0, 1, 2, \dots$) such that
 1. $\Phi(a) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi_{m,n}(a)$ where $a \in \mathcal{A}$ and the series in norm convergence.
 2. $\Phi_{m,n}(za) = z^m \bar{z}^n \Phi_{m,n}(a)$ where $z \in \mathbb{C}$ and $a \in \mathcal{A}$. The map $\Phi_{m,n}(\cdot)$ with the above property is called (m, n) -mixed homogeneous.
- *Theorem ([4])* Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be (m, n) -mixed homogeneous map with $m + n > 0$. Then, there is a completely positive map $\phi : \mathcal{A}_1 \times \dots \times \mathcal{A}_{m+n} \rightarrow \mathcal{B}$ with $\mathcal{A}_k \equiv \mathcal{A}$ $k = 1, \dots, m + n$ such that

$$\Phi(a) = \phi(a, \cdot, \cdot, \cdot, a) \quad (90)$$

where $a \in \mathcal{A}$ and $\phi(a_1, \dots, a_{m+n})$ is multilinear in (a_1, \dots, a_m) and multi-conjugate-linear in $(a_{m+1}, \dots, a_{m+n})$.

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